A Fast Recursive Highly Dispersive Absorbing Boundary Condition Using Time Domain Diakoptics and Laguerre Polynomials

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Abstract—**The contribution presents a new implementation of the absorbing boundary condition for structures where traditional ABC's perform poorly because of high dispersion. The proposed approach consists in the application of time domain diakoptics combined with the expansion of the impulse response into series of Laguerre polynomials. The application of Laguerre polynomials allows fast recursive calculation of the convolutions required by diakoptics. The numerical cost associated with this approach is significantly lower than in currently used time diakoptics schemes.**

I. Introduction

TIME DOMAIN numerical methods using finite differencing scheme in the modeling of uniform infinite structures requires the proper termination of computational space in the plane where no physical boundary exists. This termination is known as the Absorbing Boundary Condition (ABC). A good ABC should not reflect impinging waves over a wide frequency range. Traditional FDTD and TLM schemes use ABC's based on the decomposition of wave equation into operators describing one way propagation and setting to zero an expression representing the wave traveling back to the interior of the structure [2]. This process results in the Mur's first and second order conditions [1].

The Mur's condition and its modifications require the knowledge of speed of the wave in the vicinity of the absorbing boundary and hence they can be used when the propagation is nondispersive or weakly dispersive. In the waveguiding structures, especially when the dispersion is large, the velocity of wave varies with frequency. If the structure is shielded and does not support TEM waves the velocity of electromagnetic waves changes from zero at cutoff to the speed of light in free space at the high frequency end. This feature implies that when dispersion is large alternative techniques have to be used.

One approach currently used for highly dispersive structures is to replace waveguide by a long section of a lossy transmission line. With a lossy line approach the return loss better than -80 dB can be obtained in the octave band at the numerical cost ranging from few hundred to few thousand memory cells and floating point operations per iteration [7]. Another way to handle the dispersion is to apply the Berenger condition [9]. Recent publications show that this approach provides much better results even near the cutoff region [10].

Alternative approach to dispersive stuctures, called the time domain diakoptics [3], [4] is also being considered by some authors. In the context of ABC's for waveguides, the time domain diakoptics consists in finding the impulse response of a section of the waveguide and using the convolution integral to predict value of the field at the ABC from the value of the field at the input of this section. This relation is written in the form

$$r(t) = f(t) * h(t) \tag{1}$$

where $*$ is the symbol for the convolution integral and $h(t)$ is the impulse response of the system, $f(t)$ is the field value at the input of the waveguide section and $r(t)$ is the required field value at the ABC. For functions sampled at time instances $i\Delta t$, the response $r(n) = r(n\Delta t)$ is evaluated by the summation

$$r(n) = \Delta t \sum_{m=1}^{n-1} f(n-m)h(m). \tag{2}$$

For the TLM method the time domain diakoptics treatment of the ABC was initially implemented by individually convolving signals at each point of discretization [4]. This approach is computationally inefficient so very recently it has been refined for the ABC in homogeneous guides by first converting the discrete representation of fields into modes and convolving each mode separately [5], [6]. However, since the impulse response of a section of a waveguide is long, the numerical cost of calculating the convolution was still high (at least a few hundred floating point operations per iteration step).

II. Low Cost Algorithm for ABC

In order to drastically reduce the numerical cost we have developed a new algorithm for homogeneous highly dispersive guides. The algorithm is based on the time domain diakoptics in the version involving decomposition into modes before convolution [5], [6] but using the expansion of the modal impulse response of a section of a waveguide into series of the Laguerre polynomials. The Laguerre polynomials lead to very fast recursive formulas for computation of convolution integrals. It has to be noted that, the recursive computation of

Manuscript received November 30, 1994. This work was supported by the Polish Committee of Scientific Research (Contract 8T11D 00108).

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IEEE Log Number 0410992.

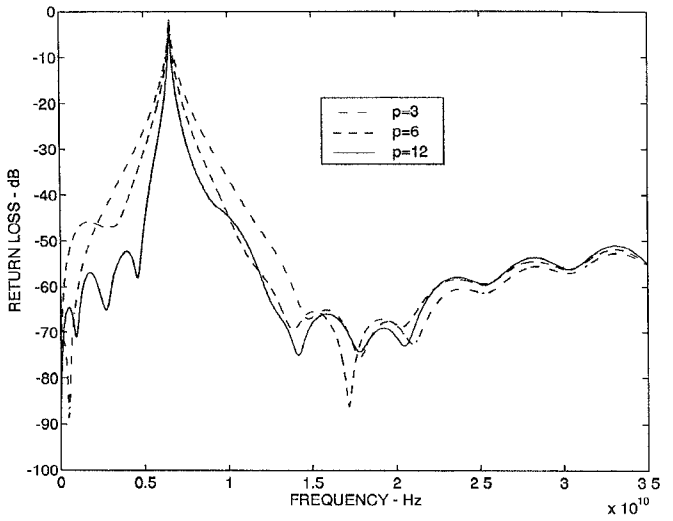

Fig. 1. The return loss of the ABC for a TE_{10} mode in a 10.16 by 22.86 mm rectangularwaveguide for increasing number of terms in expansion (3) and $\xi = 0.9$.

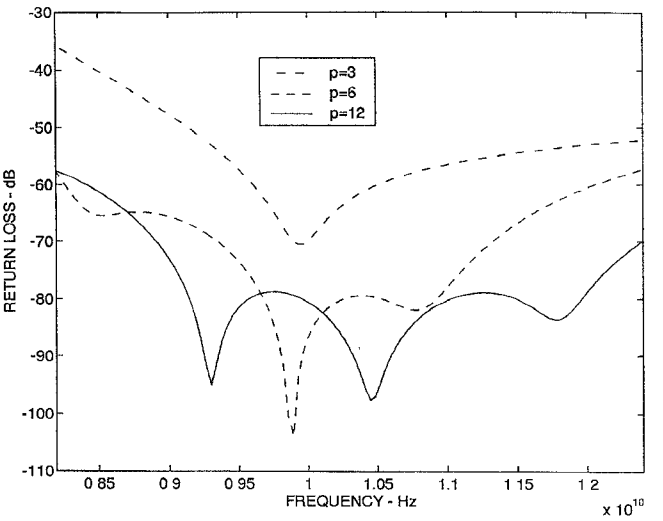

Fig. 2. The return loss of the ABC for single TE_{10} mode operation in a 10 16 × 22 86 mm rectangular waveguide for increasing number of terms in expansion (3) and $\xi = 0.85$

convolution can be achieved also with other bases including damped exponentials (Prony's method) or Kautz-Laguerre polynomials. In particular Prony's method has been used previously [2] for time domain analysis of wave propagation in dispersive media and the modeling of on surface impedance boundary conditions. As long as the convolution can be computed recursively or evaluated at low cost, the choice of the expansion basis is somewhat arbitrary and for a particular application various function representations can be tested in order to find an optimum solution. Because of the space limit the problem of the optimal impulse response representation can not be discussed in this letter. Generally, Laguerre polynomials are recommended for the description of systems with long impulse response [8]. An advantage of this choice over the Prony's method may be the fact that Prony's method yields a nonlinear optimization problem and in order to avoid computational problems it is most often implemented in a suboptimal form (Least Squares or SVD), while approximation with Laguerre polynomials leads to a much simpler and optimal formulas.

The expansion of the impulse response in the Laguerre basis can be written in the following form:

$$h(t) = \sum_{p=1}^{\infty} c_p L_p(t) \tag{3}$$

where $L_p(t)$ denotes the Laguerre polynomial of the order p. The z-transform of the Laguerre polynomials is given by

$$\mathcal{L}_p(z,\xi) = \sqrt{1-\xi^2}\frac{z}{z-\xi}\left[\frac{1-\xi z}{z-\xi}\right]^{p-1} \tag{4}$$

for a parameter $\xi \in (0,1)$. Since the Laguerre polynomials are orthogonal and the impulse response for the homogeneous section of a cylindrical guide is known in the analytical form [6], the value of the expansion coefficients c_p can be found by taking the inner product of $h(t)$ with each expansion function. This leads to

$$c_p = \sum_{m=1}^{\infty} h(m) L_p(m). \tag{5}$$

In practice the finite summation is used in the above expression. With finite number of terms the parameter $\xi \in (0,1)$ provides an additional degree of freedom allowing one to minimize the approximation error for a given approximation order. If the impulse response is not explicitly given, the expansion coefficients can be calculated by means of the least squares technique form the samples of input and output signal. This does not give an overhead on the simulation time because the computations can be carried out off-line prior to simulation of an actual structure.

Using representation (3), the convolution (1) becomes

$$r(t) = \sum_{p=1}^{\infty} c_p f(t) * L_p(t). \tag{6}$$

The most important feature of representation (6) is that, unlike in (1), the convolution terms can be computed recursively. Denoting $y_p(t) = f(t) * L_p(t)$ we have the following recursion formula for $y_p(t)$ at the time instant $t = m\Delta t$

$$y_1(m) = \xi y_1(m-1) + \sqrt{1-\xi^2} f(m) \tag{7}$$

$$y_p(m) = x_p(m) - \xi x_p(m+1) \quad p = 2.3, \ldots \tag{8}$$

with

$$x_p(m+1) = \xi x_p(m) + y_{p-1}(m) \quad p = 2,3,\ldots. \tag{9}$$

The above formulas can be found in [8], but it has to be noted that that the reference contains a few typographical errors.

The recursive calculation of (6) is extremely fast and requires only few additional memory locations. If the series (3) is truncated at $p = P$ then the cost of calculating the convolution is $6P-1$ and memory required is $3P$ (per each mode).

The concept described above was implemented for TE_{10} mode in a 10.16×22.86 mm rectangular waveguide. The guide was excited with a wideband Gaussian (0–60 GHz) pulse and the excitation and signal sampling points were located ten cells from ABC. In order to allow the low speed wave components to reach the ABC the simulation continued for 8196 time steps. Fig. 1 shows the return loss for an increasing number of expansion terms and $\xi = 0.9$. The choice of parameter ξ can be optimized as discussed in [8] but in our computation we found that $\xi = 0.9$ provides satisfactory results. It is seen that even for the lowest approximation order the results are good as the return loss is lower than -50 dB over a very wide frequency band. The worst performance is seen near the cutoff frequency. The performance of ABC can be improved if one restricts the frequency range of the excitation signal. For instance, Fig. 2 shows the performance of Laguerre ABC for a practical case of the single TE_{10} mode operation (8.2–12.4 GHz). Here, $P = 3$ gives the return loss better than -40 dB and by increasing the number of Laguerre terms from 3 to 6 the performance improves on average by 20 dB in the whole frequency band. Since P is small, the numerical cost of implementing the proposed ABC is low. The overhead is marginal as the calculation of response requires only six extra floating point operations per expansion term. It has to be noted that this cost is a few orders of magnitude lower than in some most recently published results [4]–[7].

REFERENCES

[1] G. Mur, "Absorbing boundary conditions for the finite-difference approximation of the time domain electromagnetic field equations," *IEEE Trans. Electromagn. Compat.*, vol. EMC-23, p. 377, 1981.

[2] K. S. Kunz and J. R. Luebbers, *The Finite Difference Time Domain Methods for Electromagnetics*. Boca Raton, FL: CRC, 1993.

[3] P. B. Johns and A. Akhtarzad, "The use of time domain diakoptics in time domain discrete models of fields," *Int. J. Numer. Methods Eng.*, vol. 18, pp. 1361–1373, 1982.

[4] W. J. R. Hoefer, "The discrete time domain Green's function of Johns Matrix—A powerful concept in transmission line modeling," *Int. J. Numer. Modeling*, vol. 2, no. 4, pp. 215–225, 1990.

[5] M. Righi, M. Mongiardo, R. Sorrentino, and W. J. Hoefer, "Efficient TLM diakoptics for separable structures," in *IEEE MTT-S Int. Microwave Symp. Dig.*, 1993, pp. 425–428.

[6] F. Moglie, T. Rozzi, P. Marcozzi, and A. Schiavoni, "A new termination condition for the application of FDTD techniques to discontinuity problems in close homogeneous waveguide," *IEEE Microwave and Guided Wave Lett.*, vol. 2, pp. 475–477, Dec. 1992.

[7] M. Dohlus, P. Thoma, and T. Weiland, "Broadband simulation of open waveguide boundaries within large frequency ranges," in *Time Domain Modeling Workshop*, Berlin, 1993.

[8] R. E. King and P. N. Paraskevopoulos, "Digital laguerre filters," *Circuit Theory Applicat.*, vol. 5, pp. 81–91, 1977.

[9] J.-P. Berenger, "A perfectly matched layer for absorption of electromagnetic waves," *J. Computational Phys.*, vol. 114, no. 1, pp. 185–200, 1994.

[10] C. E. Reuter, R. M. Joseph, E. T. Thiele, D. S. Katz, and A. Taflove, "Ultrawideband absorbing boundary condition for termination of waveguiding structures in FD-TD simulations," *IEEE Microwave and Guided Wave Lett.*, vol. 4, pp. 344–346, Oct. 1994.